

Applicative Bisimulations for Delimited-Control Operators

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Abstract. We develop a behavioral theory for the untyped call-by-value λ -calculus extended with the delimited-control operators *shift* and *reset*. For this calculus, we discuss the possible observable behaviors and we define an applicative bisimilarity that characterizes contextual equivalence. We then compare the applicative bisimilarity and the CPS equivalence, a relation on terms often used in studies of control operators. In the process, we illustrate how bisimilarity can be used to prove equivalence of terms with delimited-control effects.

1 Introduction

Morris-style contextual equivalence [20] is usually regarded as the most natural behavioral equivalence for functional languages based on λ -calculi. Roughly, two terms are equivalent if we can exchange one for the other in a bigger program without affecting its behavior (i.e., whether it terminates or not). The quantification over program contexts makes contextual equivalence hard to use in practice and, therefore, it is common to look for more effective characterizations of this relation. One approach is to rely on coinduction, by searching for an appropriate notion of *bisimulation*. The bisimulation has to be defined in such a way that its resulting behavioral equivalence, called *bisimilarity*, is *sound* and *complete* with respect to contextual equivalence (i.e., it is included and contains contextual equivalence, respectively).

The problem of finding a sound and complete bisimilarity in the λ -calculus has been well studied and usually leads to the definition of an *applicative* bisimilarity [1,11,10] (or, more recently, *environmental* bisimilarity [22]). The situation is more complex in λ -calculi extended with *control operators* for first-class continuations—so far, only a few works have been conducted on the behavioral theory of such calculi. A first step can be found for the $\lambda\mu$ -calculus (a calculus that mimics abortive control operators such as *call/cc* [21]) in [3] and [8], where it is proved that the definition of contextual equivalence can be slightly simplified by quantifying over evaluation contexts only; such a result is usually called a *context lemma*. In [24], Støvring and Lassen define an *eager normal form bisimilarity* (based on the notion of Lévy-Longo tree equivalence) [14,15,16] which is sound for the $\lambda\mu$ -calculus, and which becomes sound and complete when a

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notion of state is added to the $\lambda\mu$ -calculus. In [18], Merro and Biasi define an applicative bisimilarity which characterizes contextual equivalence in the *CPS calculus* [25], a minimal calculus which models the control features of functional languages with imperative jumps. As for the λ -calculus extended with control only, however, no sound and complete bisimilarities have been defined.

In this article, we present a sound and complete applicative bisimilarity for a λ -calculus extended with Danvy and Filinski’s static delimited-control operators *shift* and *reset* [7]. In contrast to abortive control operators, delimited-control operators allow to delimit access to the current continuation and to compose continuations. The operators *shift* and *reset* were introduced as a direct-style realization of the traditional success/failure continuation model of backtracking otherwise expressible only in continuation-passing style. The numerous theoretical and practical applications of *shift* and *reset* (see, e.g., [5] for an extensive list) include the seminal result by Filinski showing that a programming language endowed with *shift* and *reset* is monadically complete [9].

The λ -calculi with static delimited-control operators have been an active research topic from the semantics as well as type- and proof-theoretic point of view (see, e.g., [5,4,2]). However, to our knowledge, no work has been carried out on the behavioral theory of such λ -calculi. In order to fill this void, we present a study of the behavioral theory of an untyped, call-by-value λ -calculus extended with *shift* and *reset* [7], called λ_S . In Section 2, we give the syntax and reduction semantics of λ_S , and discuss the possible observable behaviors for the calculus. In Section 3, we define an applicative bisimilarity, based on a labelled transition semantics, and prove it characterizes contextual equivalence, using an adaptation of Howe’s congruence proof method [11]. As a byproduct, we also prove a context lemma for λ_S . In Section 4, we study the relationship between applicative bisimilarity and an equivalence based on translation into continuation-passing style (CPS), a relation often used in works on control operators and CPS. In the process, we show how applicative bisimilarity can be used to prove equivalence of terms. Section 5 concludes the article and gives ideas for future work. The appendices contain the proofs missing from the body of the article.

2 The Language λ_S

In this section, we present the syntax, reduction semantics, and contextual equivalence of the language λ_S used throughout this article.

2.1 Syntax

The language λ_S extends the call-by-value λ -calculus with the delimited-control operators *shift* and *reset* [7]. We assume we have a set of term variables, ranged over by x and k . We use two metavariables to distinguish term variables bound with a λ -abstraction from variables bound with a *shift*; we believe such distinction helps to understand examples and reduction rules. The syntax of terms and

values is given by the following grammars:

Terms: $t ::= x \mid \lambda x.t \mid t t \mid Sk.t \mid \langle t \rangle$
 Values: $v ::= \lambda x.t$

The operator *shift* ($Sk.t$) is a capture operator, the extent of which is determined by the delimiter *reset* ($\langle \cdot \rangle$). A λ -abstraction $\lambda x.t$ binds x in t and a shift construct $Sk.t$ binds k in t ; terms are equated up to α -conversion of their bound variables. The set of free variables of t is written $\text{fv}(t)$; a term is *closed* if it does not contain any free variable. Because we work mostly with closed terms, we consider only λ -abstractions as values.

We distinguish several kinds of contexts, defined below, which all can be seen as terms with a hole.

Pure evaluation contexts: $E ::= \square \mid v E \mid E t$
 Evaluation contexts: $F ::= \square \mid v F \mid F t \mid \langle F \rangle$
 Contexts: $C ::= \square \mid \lambda x.C \mid t C \mid C t \mid Sk.C \mid \langle C \rangle$

Regular contexts are ranged over by C . The pure evaluation contexts¹ (abbreviated as pure contexts), ranged over by E , represent delimited continuations and can be captured by the shift operator. The call-by-value evaluation contexts, ranged over by F , represent arbitrary continuations and encode the chosen reduction strategy. Following the correspondence between evaluation contexts of the reduction semantics and control stacks of the abstract machine for shift and reset, established by Biernacka et al. [5], we interpret contexts inside-out, i.e., \square stands for the empty context, $v E$ represents the “term with a hole” $E[v []]$, $E t$ represents $E[[] t]$, $\langle F \rangle$ represents $F[\langle [] \rangle]$, etc. (This choice does not affect the results presented in this article in any way.) Filling a context C (respectively E , F) with a term t produces a term, written $C[t]$ (respectively $E[t]$, $F[t]$); the free variables of t can be captured in the process. A context is *closed* if it contains only closed terms.

2.2 Reduction Semantics

Let us first briefly describe the intuitive semantics of shift and reset by means of an example written in SML using Filinski’s implementation of shift and reset [9].

Example 1. The following function copies a list [6] (the SML expression `shift (fn k => t)` corresponds to $Sk.t$ and `reset (fn () => t)` corresponds to $\langle t \rangle$):

```
fun copy xs =
  let fun visit nil = nil
      | visit (x::xs) = visit (shift (fn k => x :: (k xs)))
  in reset (fn () => visit xs) end
```

This program illustrates the main ideas of programming with shift and reset:

¹ This terminology comes from Kameyama (e.g., in [12]).

- Reset delimits continuations. Control effects are local to `copy`.
- Shift captures delimited continuations. Each, but last, recursive call to `visit` abstracts the continuation `fn v => reset (fn () => visit v)` and binds it to `k`.
- Captured continuations are statically composed. When applied in the expression `k xs`, the captured continuation becomes the current delimited continuation that is isolated from the rest of the program by a control delimiter—witness the `reset` expression in the captured continuation.

Formally, the call-by-value reduction semantics of λ_S is defined by the following rules, where $t\{v/x\}$ is the usual capture-avoiding substitution of v for x in t :

$$\begin{array}{ll}
(\beta_v) & F[(\lambda x.t) v] \rightarrow_v F[t\{v/x\}] \\
(shift) & F[\langle E[\mathcal{S}k.t] \rangle] \rightarrow_v F[\langle t\{\lambda x.\langle E[x] \rangle/k \} \rangle] \text{ with } x \notin \text{fv}(E) \\
(reset) & F[\langle v \rangle] \rightarrow_v F[v]
\end{array}$$

The term $(\lambda x.t) v$ is the usual call-by-value redex for β -reduction (rule (β_v)). The operator $\mathcal{S}k.t$ captures its surrounding context E up to the dynamically nearest enclosing `reset`, and substitutes $\lambda x.\langle E[x] \rangle$ for k in t (rule $(shift)$). If a `reset` is enclosing a value, then it has no purpose as a delimiter for a potential capture, and it can be safely removed (rule $(reset)$). All these reductions may occur within a metalevel context F . The chosen call-by-value evaluation strategy is encoded in the grammar of the evaluation contexts.

Example 2. Let $i = \lambda x.x$ and $\omega = \lambda x.xx$. We present the sequence of reductions initiated by $\langle ((\mathcal{S}k_1.i (k_1 i)) \mathcal{S}k_2.\omega) (\omega \omega) \rangle$. The term $\mathcal{S}k_1.i (k_1 i)$ is within the pure context $E = (\square (\omega \omega)) \mathcal{S}k_2.\omega$ (remember that we represent contexts inside-out), enclosed in a delimiter $\langle \cdot \rangle$, so E is captured according to rule $(shift)$.

$$\langle ((\mathcal{S}k_1.i (k_1 i)) \mathcal{S}k_2.\omega) (\omega \omega) \rangle \rightarrow_v \langle i ((\lambda x.\langle (x \mathcal{S}k_2.\omega) (\omega \omega) \rangle) i) \rangle$$

The role of `reset` in $\lambda x.\langle E[x] \rangle$ becomes clearer after reduction of the (β_v) -redex $(\lambda x.\langle E[x] \rangle) i$.

$$\langle i ((\lambda x.\langle (x \mathcal{S}k_2.\omega) (\omega \omega) \rangle) i) \rangle \rightarrow_v \langle i \langle (i \mathcal{S}k_2.\omega) (\omega \omega) \rangle \rangle$$

When the captured context E is reactivated, it is not *merged* with the context $i \square$, but *composed* thanks to the `reset` enclosing E . As a result, the capture triggered by $\mathcal{S}k_2.\omega$ leaves the term i outside the first enclosing `reset` untouched.

$$\langle i \langle (i \mathcal{S}k_2.\omega) (\omega \omega) \rangle \rangle \rightarrow_v \langle i \langle \omega \rangle \rangle$$

Because k_2 does not occur in ω , the context $i (\square (\omega \omega))$ is discarded when captured by $\mathcal{S}k_2.\omega$. Finally, we remove the useless delimiter $\langle i \langle \omega \rangle \rangle \rightarrow_v \langle i \omega \rangle$ with rule $(reset)$, and we then (β_v) -reduce and remove the last delimiter $\langle i \omega \rangle \rightarrow_v \langle \omega \rangle \rightarrow_v \omega$. Note that, while the reduction strategy is call-by-value, some function arguments are not evaluated, like the non-terminating term $\omega \omega$ in this example.

There exist terms which are not values and which cannot be reduced any further; these are called *stuck terms*.

Definition 1. A closed term t is stuck if t is not a value and $t \not\rightarrow_v$.

For example, the term $E[\mathcal{S}k.t]$ is stuck because there is no enclosing reset; the capture of E by the shift operator cannot be triggered. In fact, closed stuck terms are easy to characterize.

Lemma 1. A closed term t is stuck iff $t = E[\mathcal{S}k.t']$ for some E , k , and t' .

We call *redexes* (ranged over by r) the terms of the form $(\lambda x.t)v$, $\langle E[\mathcal{S}k.t] \rangle$, and $\langle v \rangle$. Thanks to the following unique-decomposition property, the reduction \rightarrow_v is deterministic.

Lemma 2. For all closed terms t , either t is a value, or it is a stuck term, or there exist a unique redex r and a unique context F such that $t = F[r]$.

Given a relation \mathcal{R} on terms, we write \mathcal{R}^* for the transitive and reflexive closure of \mathcal{R} . We define the evaluation relation of λ_S as follows.

Definition 2. We write $t \Downarrow_v t'$ if $t \rightarrow_v^* t'$ and $t' \not\rightarrow_v$.

The result of the evaluation of a closed term, if it exists, is either a value or a stuck term. If a term t admits an infinite reduction sequence, we say it *diverges*, written $t \Uparrow_v$. In the rest of the article, we use extensively $\Omega = (\lambda x.x x) (\lambda x.x x)$ as an example of such a term.

2.3 Contextual Equivalence

In this section, we discuss the possible definitions of a Morris-style contextual equivalence for the calculus λ_S . As usual, the idea is to express that two terms are equivalent iff they cannot be distinguished when put in an arbitrary context. The question is then what kind of behavior we want to observe. As in the regular λ -calculus we could observe only termination (i.e., does a term reduce to a value or not), leading to the following relation.

Definition 3. Let t_0, t_1 be closed terms. We write $t_0 \approx_c^1 t_1$ if for all closed C , $C[t_0] \Downarrow_v v_0$ implies $C[t_1] \Downarrow_v v_1$, and conversely for $C[t_1]$.

This definition does not mention stuck terms; as a result, they can be equated with diverging terms. For example, let $t_0 = (\mathcal{S}k.k \lambda x.x) \Omega$, $t_1 = \Omega$, and C be a closed context. If $C[t_0] \Downarrow_v v_0$, then we can prove that for all closed t , there exists v such that $C[t] \Downarrow_v v$ (roughly, because t is never evaluated; see Lemma 8 in Appendix A). In particular, we have $C[t_1] \Downarrow_v v_1$. Hence, we have $t_0 \approx_c^1 t_1$.

A more fine-grained analysis is possible, by observing stuck terms.

Definition 4. Let t_0, t_1 be closed terms. We write $t_0 \approx_c^2 t_1$ if for all closed C ,

- $C[t_0] \Downarrow_v v_0$ implies $C[t_1] \Downarrow_v v_1$;
- $C[t_0] \Downarrow_v t'_0$, where t'_0 is stuck, implies $C[t_1] \Downarrow_v t'_1$, with t'_1 stuck as well;

and conversely for $C[t_1]$.

The relation \approx_c^2 distinguishes the terms t_0 and t_1 defined above. We believe \approx_c^2 is more interesting because it gives more information on the behavior of terms; consequently, we use it as the contextual equivalence for λ_S . Henceforth, we simply write \approx_c for \approx_c^2 .

The relation \approx_c , like the other equivalences on terms defined in this article, can be extended to open terms in the following way.

Definition 5. *Let \mathcal{R} be a relation on closed terms. The open extension of \mathcal{R} , written \mathcal{R}° , is defined on open terms as: we write $t_0 \mathcal{R}^\circ t_1$ if for every substitution σ which closes t_0 and t_1 , $t_0\sigma \mathcal{R} t_1\sigma$ holds.*

Remark 1. Contextual equivalence can be defined directly on open terms by requiring that the context C binds the free variables of the related terms. The resulting relation would be equal to \approx_c° [10].

3 Bisimilarity for λ_S

In this section, we define an applicative bisimilarity and prove it equal to contextual equivalence.

3.1 Labelled Transition System

To define the bisimilarity for λ_S , we propose a labelled transition system (LTS), where the possible interactions of a term with its environment are encoded in the labels. Figure 1 defines a LTS $t_0 \xrightarrow{\alpha} t_1$ with three kinds of transitions. An *internal action* $t \xrightarrow{\tau} t'$ is an evolution from t to t' without any help from the surrounding context; it corresponds to a reduction step from t to t' . The transition $v_0 \xrightarrow{v_1} t$ expresses the fact that v_0 needs to be applied to another value v_1 to evolve, reducing to t . Finally, the transition $t \xrightarrow{E} t'$ means that t is stuck, and when t is put in a context E enclosed in a reset, the capture can be triggered, the result of which being t' .

Most rules for internal actions (Fig. 1) are straightforward; the rules (β_v) and (reset) mimic the corresponding reduction rules, and the compositional rules (right_τ) , (left_τ) , and $(\langle \cdot \rangle_\tau)$ allow internal actions to happen within any evaluation context. The rule $(\langle \cdot \rangle_S)$ for context capture is explained later. Rule (val) defines the only possible transition for values. Note that while both rules (β_v) and (val) encode β -reduction, they are quite different in nature; in the former, the term $(\lambda x.t) v$ can evolve by itself, without any help from the surrounding context, while the latter expresses the possibility for $\lambda x.t$ to evolve only if a value v is provided by the environment.

The rules for context capture are built following the principles of elementary semantics developed in [17]. The label of the transition $t \xrightarrow{E} t'$ contains what the environment needs to provide (a context E , but also an enclosing reset, left implicit) for the stuck term t to reduce to t' . Hence, the transition $t \xrightarrow{E} t'$

$\frac{}{(\lambda x.t) v \xrightarrow{\tau} t\{v/x\}} (\beta_v)$	$\frac{}{\langle v \rangle \xrightarrow{\tau} v} (\text{reset})$	$\frac{t_0 \xrightarrow{\tau} t'_0}{t_0 t_1 \xrightarrow{\tau} t'_0 t_1} (\text{left}_\tau)$
$\frac{t \xrightarrow{\tau} t'}{v t \xrightarrow{\tau} v t'} (\text{right}_\tau)$	$\frac{t \xrightarrow{\tau} t'}{\langle t \rangle \xrightarrow{\tau} \langle t' \rangle} (\langle \cdot \rangle_\tau)$	$\frac{t \xrightarrow{\square} t'}{\langle t \rangle \xrightarrow{\tau} t'} (\langle \cdot \rangle_s)$
$\frac{}{\lambda x.t \xrightarrow{v} t\{v/x\}} (\text{val})$		
$\frac{x \notin \text{fv}(E)}{Sk.t \xrightarrow{E} \langle t\{\lambda x.\langle E[x] \rangle/k\} \rangle} (\text{shift})$	$\frac{t_0 \xrightarrow{E t_1} t'_0}{t_0 t_1 \xrightarrow{E} t'_0} (\text{left}_s)$	$\frac{t \xrightarrow{v E} t'}{v t \xrightarrow{E} t'} (\text{right}_s)$

Fig. 1: Labelled Transition System

means that we have $\langle E[t] \rangle \xrightarrow{\tau} t'$ by context capture. For example, in the rule (shift), the result of the capture of E by $Sk.t$ is $\langle t\{\lambda x.\langle E[x] \rangle/k\} \rangle$.

In rule (left_s), we want to know the result of the capture of E by the term $t_0 t_1$, assuming t_0 contains an operator shift. Under this hypothesis, the capture of E by $t_0 t_1$ comes from the capture of $E t_1$ by t_0 . Therefore, as premise of the rule (left_s), we check that t_0 is able to capture $E t_1$, and the result t'_0 of this transition is exactly the result we want for the capture of E by $t_0 t_1$. The rule (right_s) follows the same pattern. Finally, a stuck term t enclosed in a reset is able to perform an internal action (rule $(\langle \cdot \rangle_s)$); we obtain the result t' of the transition $\langle t \rangle \xrightarrow{\tau} t'$ by letting t capture the empty context, i.e., by considering the transition $t \xrightarrow{\square} t'$.

Example 3. With the same notations as in Example 2, we illustrate how the LTS handles capture by considering the transition from $\langle (i Sk.\omega) (\omega \omega) \rangle$.

$$\begin{array}{c}
\frac{}{Sk.\omega \xrightarrow{i (\square (\omega \omega))} \langle \omega \rangle} (\text{shift}) \\
\frac{Sk.\omega \xrightarrow{i (\square (\omega \omega))} \langle \omega \rangle}{i Sk.\omega \xrightarrow{\square (\omega \omega)} \langle \omega \rangle} (\text{right}_s) \\
\frac{i Sk.\omega \xrightarrow{\square (\omega \omega)} \langle \omega \rangle}{(i Sk.\omega) (\omega \omega) \xrightarrow{\square} \langle \omega \rangle} (\text{left}_s) \\
\frac{(i Sk.\omega) (\omega \omega) \xrightarrow{\square} \langle \omega \rangle}{\langle (i Sk.\omega) (\omega \omega) \rangle \xrightarrow{\tau} \langle \omega \rangle} (\langle \cdot \rangle_s)
\end{array}$$

Reading the tree from bottom to top, we see that the rules $(\langle \cdot \rangle_s)$, (left_s), and (right_s) build the captured context in the label by deconstructing the initial term. Indeed, the rule $(\langle \cdot \rangle_s)$ removes the outermost reset, and initiates the context in the label with \square . The rules (left_s) and (right_s) then successively remove the outermost application and store it in the context. The process continues until a shift operator is found; then we know the captured context is completed, and the rule (shift) computes the result of the capture. This result is then simply propagated from top to bottom by the other rules.

The LTS corresponds to the reduction semantics and exhibits the observable terms (values and stuck terms) of the language in the following way.

Lemma 3. *The following hold:*

- We have $\xrightarrow{\tau} = \rightarrow_v$.
- If $t \xrightarrow{E} t'$, then t is a stuck term, and $\langle E[t] \rangle \xrightarrow{\tau} t'$.
- If $t \xrightarrow{v} t'$, then t is a value, and $t v \xrightarrow{\tau} t'$.

3.2 Applicative Bisimilarity

We now define the notion of applicative bisimilarity for λ_S . We write \Rightarrow for the reflexive and transitive closure of $\xrightarrow{\tau}$. We define the weak delay² transition $\xRightarrow{\alpha}$ as \Rightarrow if $\alpha = \tau$ and as $\xRightarrow{\alpha}$ otherwise. The definition of the (weak delay) bisimilarity is then straightforward.

Definition 6. *A relation \mathcal{R} on closed terms is an applicative simulation if $t_0 \mathcal{R} t_1$ implies that for all $t_0 \xRightarrow{\alpha} t'_0$, there exists t'_1 such that $t_1 \xRightarrow{\alpha} t'_1$ and $t'_0 \mathcal{R} t'_1$.*

A relation \mathcal{R} on closed terms is an applicative bisimulation if \mathcal{R} and \mathcal{R}^{-1} are simulations. Applicative bisimilarity \approx is the largest applicative bisimulation.

In words, two terms are equivalent if any transition from one is matched by a weak transition with the same label from the other. As in the λ -calculus [1,10], it is not mandatory to test the internal steps when proving that two terms are bisimilar, because of the following result.

Lemma 4. *If $t \xrightarrow{\tau} t'$ (respectively $t \Downarrow_v t'$) then $t \approx t'$.*

Lemma 4 holds because $\{(t, t'), t \xrightarrow{\tau} t'\}$ is an applicative bisimulation. Consequently, applicative bisimulation can be defined in terms of big-step transitions as follows.

Definition 7. *A relation \mathcal{R} on closed terms is a big-step applicative simulation if $t_0 \mathcal{R} t_1$ implies that for all $t_0 \xRightarrow{\alpha} t'_0$ with $\alpha \neq \tau$, there exists t'_1 such that $t_1 \xRightarrow{\alpha} t'_1$ and $t'_0 \mathcal{R} t'_1$.*

A relation \mathcal{R} on closed terms is a big-step applicative bisimulation if \mathcal{R} and \mathcal{R}^{-1} are big-step applicative simulations. Big-step applicative bisimilarity \approx is the largest big-step applicative bisimulation.

Henceforth, we drop the adjective “applicative” and refer to the two kinds of relations simply as “bisimulation” and “big-step bisimulation”.

Lemma 5. *We have $\approx = \approx$.*

The proof is by showing that \approx is a big step bisimulation, and that \approx is a bisimulation (using a variant of Lemma 4 involving \approx). As a result, if \mathcal{R} is a big-step bisimulation, then $\mathcal{R} \subseteq \approx \subseteq \approx$. We work with both styles (small-step and big-step), depending on which one is easier to use in a given proof.

Example 4. Assuming we add lists and recursion to the calculus, we informally prove that the function `copy` defined in Example 1 is bisimilar to its effect-free variant, defined below.

² where internal steps are allowed before, but not after a visible action


```

fun copy2 nil = nil
  | copy2 (x::xs) = x::(copy2 xs)

```

To this end, we define the relations (where we let l range over lists, and e over their elements)

$$\begin{aligned}
\mathcal{R}_1 &= \{(\langle e_1 :: \langle e_2 :: \dots \langle e_n :: \langle \text{visit } l \rangle \rangle \rangle), e_1 :: (e_2 :: \dots e_n :: (\text{copy2 } l))\}) \\
\mathcal{R}_2 &= \{(\langle e_1 :: \langle e_2 :: \dots \langle e_n :: \langle l \rangle \rangle \rangle), e_1 :: (e_2 :: \dots e_n :: l)\})
\end{aligned}$$

and we prove that $\mathcal{R}_1 \cup \mathcal{R}_2 \cup \{(l, l)\}$ is a bisimulation. First, let $t_0 \mathcal{R}_1 t_1$. If l is empty, then both $\text{visit } l$ and $\text{copy2 } l$ reduce to the empty list, and we obtain two terms related by \mathcal{R}_2 . Otherwise, we have $l = e_{n+1} :: l'$, $\langle \text{visit } l \rangle$ reduces to $\langle e_{n+1} :: \langle \text{visit } l' \rangle \rangle$, $\text{copy2 } l$ reduces to $e_{n+1} :: (\text{copy2 } l')$, and therefore t_0 and t_1 reduce to terms that are still in \mathcal{R}_1 . Now, consider $t_0 \mathcal{R}_2 t_1$; the transition from t_0 removes the delimiter surrounding l , giving a term related by \mathcal{R}_2 to t_1 if there are still some delimiters left, or equal to t_1 if all the delimiters are removed. Finally, two identical lists are clearly bisimilar.

3.3 Soundness

To prove soundness of \approx w.r.t. contextual equivalence, we show that \approx is a congruence using *Howe's method*, a well-known congruence proof method initially developed for the λ -calculus [11,10]. We briefly sketch the method and explain how we apply it to \approx ; the complete proof can be found in Appendix B.2.

The idea of the method is as follows: first, prove some basic properties of *Howe's closure* \approx^\bullet , a relation which contains \approx and is a congruence by construction. Then, prove a simulation-like property for \approx^\bullet . From this result, prove that \approx^\bullet and \approx coincide on closed terms. Because \approx^\bullet is a congruence, it shows that \approx is a congruence as well. The definition of \approx^\bullet relies on the notion of *compatible refinement*; given a relation \mathcal{R} on open terms, the compatible refinement $\widehat{\mathcal{R}}$ relates two terms iff they have the same outermost operator and their immediate subterms are related by \mathcal{R} . Formally, it is inductively defined by the following rules.

$$\frac{}{x \widehat{\mathcal{R}} x} \quad \frac{t_0 \mathcal{R} t_1}{\lambda x. t_0 \widehat{\mathcal{R}} \lambda x. t_1} \quad \frac{t_0 \mathcal{R} t_1 \quad t'_0 \mathcal{R} t'_1}{t_0 t'_0 \widehat{\mathcal{R}} t_1 t'_1} \quad \frac{t_0 \mathcal{R} t_1}{Sk. t_0 \widehat{\mathcal{R}} Sk. t_1} \quad \frac{t_0 \mathcal{R} t_1}{\langle t_0 \rangle \widehat{\mathcal{R}} \langle t_1 \rangle}$$

Howe's closure \approx^\bullet is inductively defined as the smallest congruence containing \approx° and closed under right composition with \approx° .

Definition 8. *Howe's closure \approx^\bullet is the smallest relation satisfying:*

$$\frac{t_0 \approx^\circ t_1}{t_0 \approx^\bullet t_1} \quad \frac{t_0 \approx^\bullet \approx^\circ t_1}{t_0 \approx^\bullet t_1} \quad \frac{t_0 \widehat{\approx^\bullet} t_1}{t_0 \approx^\bullet t_1}$$

By construction, \approx^\bullet is a congruence (by the third rule of the definition), and composing on the right with \approx° gives some transitivity properties to \approx^\bullet . In particular, it helps in proving the following classical results (see [10] for the proofs).

Lemma 6 (Basic properties of \approx^\bullet). *The following hold:*

- For all t_0, t_1, v_0 , and v_1 , $t_0 \approx^\bullet t_1$ and $v_0 \approx^\bullet v_1$ implies $t_0\{v_0/x\} \approx^\bullet t_1\{v_1/x\}$.
- The relation $(\approx^\bullet)^*$ is symmetric.

The first item states that \approx^\bullet is substitutive. This property helps in establishing the simulation-like property of \approx^\bullet (second step of the method). Let $(\approx^\bullet)^c$ be the restriction of \approx^\bullet to closed terms. We cannot prove directly that $(\approx^\bullet)^c$ is a bisimulation, so we prove a stronger result instead. We extend \approx^\bullet to labels, by defining $E \approx^\bullet E'$ as the smallest congruence extending \approx^\bullet with the relation $\square \approx^\bullet \square$, and by adding the relation $\tau \approx^\bullet \tau$.

Lemma 7 (Simulation-like property). *If $t_0 (\approx^\bullet)^c t_1$ and $t_0 \xrightarrow{\alpha} t'_0$, then for all $\alpha (\approx^\bullet)^c \alpha'$, there exists t'_1 such that $t_1 \xrightarrow{\alpha'} t'_1$ and $t'_0 (\approx^\bullet)^c t'_1$.*

Using Lemma 7 and the fact that $((\approx^\bullet)^c)^*$ is symmetric (by the second item of Lemma 6), we can prove that $((\approx^\bullet)^c)^*$ is a bisimulation. Therefore, we have $((\approx^\bullet)^c)^* \subseteq \approx$, and because $\approx \subseteq ((\approx^\bullet)^c)^* \subseteq ((\approx^\bullet)^c)^*$ holds by construction, we can deduce $\approx = ((\approx^\bullet)^c)^*$. Because $(\approx^\bullet)^c$ is a congruence, we have the following result.

Theorem 1. *The relation \approx is a congruence.*

As a corollary, \approx is sound w.r.t. contextual equivalence.

Theorem 2. *We have $\approx \subseteq \approx_c$.*

3.4 Completeness and Context Lemma

In this section, we prove that \approx is complete w.r.t. \approx_c . To this end, we use an auxiliary relation $\dot{\approx}_c$, defined below, which refines contextual equivalence by testing terms with evaluation contexts only. While proving completeness, we also prove $\dot{\approx}_c = \approx_c$, which means that testing with evaluation contexts is as discriminative as testing with any contexts. Such a simplification result is similar to Milner's context lemma [19].

Definition 9. *Let t_0, t_1 be closed terms. We write $t_0 \dot{\approx}_c t_1$ if for all closed F ,*

- $F[t_0] \Downarrow_v v_0$ implies $F[t_1] \Downarrow_v v_1$;
- $F[t_0] \Downarrow_v t'_0$, where t'_0 is stuck, implies $F[t_1] \Downarrow_v t'_1$, with t'_1 stuck as well;

and conversely for $F[t_1]$.

Clearly we have $\approx_c \subseteq \dot{\approx}_c$ by definition. The relation \approx is complete w.r.t. $\dot{\approx}_c$.

Theorem 3. *We have $\dot{\approx}_c \subseteq \approx$.*

The proof of Theorem 3 is the same as in λ -calculus [10]; we prove that $\dot{\approx}_c$ is a big-step bisimulation, using Lemmas 3, 4, and Theorem 2. The complete proof can be found in Appendix B.3. We can now prove that all the relations defined so far coincide.

Theorem 4. *We have $\approx_c = \dot{\approx}_c = \approx$.*

Indeed, we have $\dot{\approx}_c \subseteq \approx$ (Theorem 3), $\approx \subseteq \approx_c$ (Theorem 2), and $\approx_c \subseteq \dot{\approx}_c$ (by definition).

$$\begin{aligned}
\overline{x} &= \lambda k_1 k_2. k_1 x k_2 \\
\overline{\lambda x. t} &= \lambda k_1 k_2. k_1 (\lambda x. \overline{t}) k_2 \\
\overline{t_0 t_1} &= \lambda k_1 k_2. \overline{t_0} (\lambda x_0 k'_2. \overline{t_1} (\lambda x_1 k'_2. x_0 x_1 k_1 k'_2) k'_2) k_2 \\
\overline{\langle t \rangle} &= \lambda k_1 k_2. \overline{t} \theta (\lambda x. k_1 x k_2) \\
\overline{S k. t} &= \lambda k_1 k_2. \overline{t} \{ (\lambda x_1 k'_1 k'_2. k_1 x_1 (\lambda x_2. k'_1 x_2 k'_2)) / k \} \theta k_2 \\
&\text{with } \theta = \lambda x k_2. k_2 x
\end{aligned}$$

Fig. 2: CPS translation

4 Relation to CPS Equivalence

In this section we study the relationship between our bisimilarity (and thus contextual equivalence) and an equivalence relation based on translating terms with shift and reset into continuation-passing style (CPS). Such an equivalence has been characterized in terms of direct-style equations by Kameyama and Hasegawa who developed an axiomatization of shift and reset [12]. We show that all but one of their axioms are validated by the bisimilarity of this article, which also provides several examples of use of the bisimilarity. We also pinpoint where the two relations differ.

4.1 Axiomatization of Delimited Continuations

The operators shift and reset have been originally defined by a translation into continuation-passing style [7] that we present in Fig. 2. Translated terms expect two continuations: the delimited continuation representing the rest of the computation up to the dynamically nearest enclosing delimiter and the meta-continuation representing the rest of the computation beyond this delimiter.

It is natural to relate any other theory of shift and reset to their definitional CPS translation. For example, the reduction rules $t \rightarrow_{\nu} t'$ given in Section 2.2 are sound w.r.t. the CPS because CPS translating t and t' yields $\beta\eta$ -convertible terms in the λ -calculus. More generally, the CPS translation for shift and reset induces the following notion of equivalence on terms:

Definition 10. *Terms t and t' are CPS equivalent if their CPS translations are $\beta\eta$ -convertible.*

In order to relate the bisimilarity of this article and the CPS equivalence, we use Kameyama and Hasegawa's axioms [12], which characterize the CPS equivalence in a sound and complete way: two terms are CPS equivalent iff one can derive their equality using the equations of Fig. 3. Kameyama and Hasegawa's axioms relate not only closed, but arbitrary terms and they assume variables as values.

4.2 Kameyama and Hasegawa's Axioms through Bisimilarity

We show that closed terms related by all the axioms except for $\mathcal{S}_{\text{elim}}$ are bisimilar. In the following, we write \mathcal{I} for the bisimulation $\{(t, t)\}$.

$(\lambda x.t) v = t\{v/x\}$	β_v	$(\lambda x.E[x]) t = E[t] \text{ if } x \notin \text{fv}(E)$	β_Ω
$\langle E[Sk.t] \rangle = \langle t\{\lambda x.\langle E[x] \rangle/k\} \rangle$	$\langle \cdot \rangle \mathcal{S}$	$\langle (\lambda x.t_0) \langle t_1 \rangle \rangle = (\lambda x.\langle t_0 \rangle) \langle t_1 \rangle$	$\langle \cdot \rangle \text{lift}$
$\langle v \rangle = v$	$\langle \cdot \rangle \text{val}$	$Sk.\langle t \rangle = Sk.t$	$\mathcal{S} \langle \cdot \rangle$
$\lambda x.v x = v \text{ if } x \notin \text{fv}(v)$	η_v	$Sk.k t = t \text{ if } k \notin \text{fv}(t)$	$\mathcal{S} \text{elim}$

Fig. 3: Axiomatization of $\lambda_{\mathcal{S}}$

Proposition 1. *We have $(\lambda x.t) v \approx t\{v/x\}$, $\langle E[Sk.t] \rangle \approx \langle t\{\lambda x.\langle E[x] \rangle/k\} \rangle$, and $\langle v \rangle \approx v$.*

Proof. These are direct consequences of the fact that $\xrightarrow{\tau} \subseteq \approx$ (Lemma 4). \square

Proposition 2. *If $x \notin \text{fv}(v)$, then $\lambda x.v x \approx v$.*

Proof. We prove that $\mathcal{R} = \{(\lambda x.(\lambda y.t) x, \lambda y.t), x \notin \text{fv}(t)\} \cup \approx$ is a bisimulation. To this end, we have to check that $\lambda x.(\lambda y.t) x \xrightarrow{v_0} (\lambda y.t) v_0$ is matched by $\lambda y.t \xrightarrow{v_0} t\{v_0/y\}$, i.e., that $(\lambda y.t) v_0 \mathcal{R} t\{v_0/y\}$ holds for all v_0 . We have $(\lambda y.t) v_0 \xrightarrow{\tau} t\{v_0/y\}$, and because $\xrightarrow{\tau} \subseteq \approx \subseteq \mathcal{R}$, we have the required result. \square

Proposition 3. *We have $Sk.\langle t \rangle \approx Sk.t$.*

Proof. Let $\mathcal{R} = \{(\langle \langle t \rangle \rangle, \langle t \rangle)\}$. We prove that $\{(Sk.\langle t \rangle, Sk.t)\} \cup \mathcal{R} \cup \mathcal{I}$ is a big-step bisimulation. The transition $Sk.\langle t \rangle \xrightarrow{E} \langle \langle t\{\lambda x.\langle E[x] \rangle/k\} \rangle \rangle$ is matched by $Sk.t \xrightarrow{E} \langle t\{\lambda x.\langle E[x] \rangle/k\} \rangle$, and conversely. Let $\langle \langle t \rangle \rangle \mathcal{R} \langle t \rangle$. It is straightforward to check that $\langle \langle t \rangle \rangle \xrightarrow{v_0} v$ iff $t \xrightarrow{v_0} v$ iff $\langle t \rangle \xrightarrow{v_0} v$. Therefore, any \xrightarrow{v} transition from $\langle \langle t \rangle \rangle$ is matched by $\langle t \rangle$, and conversely. If $\langle t \rangle \xrightarrow{\tau} t'$, then t' is a value or $t' = \langle t'' \rangle$ for some t'' ; consequently, neither $\langle t \rangle$ nor $\langle \langle t \rangle \rangle$ can perform a \xrightarrow{E} transition. \square

Proposition 4. *We have $\langle (\lambda x.t_0) \langle t_1 \rangle \rangle \approx (\lambda x.\langle t_0 \rangle) \langle t_1 \rangle$.*

Proof. We prove that $\{(\langle (\lambda x.t_0) \langle t_1 \rangle \rangle, (\lambda x.\langle t_0 \rangle) \langle t_1 \rangle)\} \cup \mathcal{I}$ is a big-step bisimulation. A transition $\langle (\lambda x.t_0) \langle t_1 \rangle \rangle \xrightarrow{\alpha} t'$ (with $\alpha \neq \tau$) is possible only if $\langle t_1 \rangle$ evaluates to some value v . In this case, we have $\langle (\lambda x.t_0) \langle t_1 \rangle \rangle \xrightarrow{\tau} \langle t_0\{v/x\} \rangle$ and $(\lambda x.\langle t_0 \rangle) \langle t_1 \rangle \xrightarrow{\tau} \langle t_0\{v/x\} \rangle$. From this, it is easy to see that $\langle (\lambda x.t_0) \langle t_1 \rangle \rangle \xrightarrow{\alpha} t'$ (with $\alpha \neq \tau$) implies $(\lambda x.\langle t_0 \rangle) \langle t_1 \rangle \xrightarrow{\alpha} t'$, and conversely. \square

Proposition 5. *If $x \notin \text{fv}(E)$, then $(\lambda x.E[x]) t \approx E[t]$.*

Proof (Sketch). The complete proof, quite technical, can be found in Appendix C. Let E_0 be such that $\text{fv}(E_0) = \emptyset$. Given two families of contexts $(E_1^i)_i, (E_2^i)_i$, we write $\sigma_i^{E_0}$ (resp. $\sigma_i^{\lambda x.E_0[x]}$) the substitution mapping k_i to $\lambda y.\langle E_1^i[E_0[E_2^i[y]]] \rangle$ (resp. $\lambda y.\langle E_1^i[(\lambda x.E_0[x]) E_2^i[y]] \rangle$). We define

$$\begin{aligned} \mathcal{R}_1 &= \{(F[(\lambda x.E_0[x]) t] \sigma_0^{\lambda x.E_0[x]} \dots \sigma_n^{\lambda x.E_0[x]}, F[E_0[t]] \sigma_0^{E_0} \dots \sigma_n^{E_0}), \\ &\quad \text{fv}(t, F) \subseteq \{k_0 \dots k_n\}\} \\ \mathcal{R}_2 &= \{(t \sigma_0^{\lambda x.E_0[x]} \dots \sigma_n^{\lambda x.E_0[x]}, t \sigma_0^{E_0} \dots \sigma_n^{E_0}), \text{fv}(t) \subseteq \{k_0 \dots k_n\}\} \end{aligned}$$

and we prove that $\mathcal{R}_1 \cup \mathcal{R}_2$ is a bisimulation. The relation \mathcal{R}_1 contains the terms related by Proposition 5. The transitions from terms in \mathcal{R}_1 give terms in \mathcal{R}_1 , except if a capture happens in t ; in this case, we obtain terms in \mathcal{R}_2 . Similarly, most transitions from terms in \mathcal{R}_2 give terms in \mathcal{R}_2 , except if a term $\lambda y. \langle E_1^i[E_0[E_2^i[y]]] \rangle$ (resp. $\lambda y. \langle E_1^i[(\lambda x. E_0[x]) E_2^i[y]] \rangle$) is applied to a value (i.e., if $t = F[k_i v]$). In this case, the β -reduction generates terms in \mathcal{R}_1 . \square

4.3 Bisimilarity and CPS Equivalence

In Section 4.2, we have considered all the axioms of Fig. 3, except $\mathcal{S} \text{ elim}$. The terms $\mathcal{S}k.k t$ (with $k \notin \text{fv}(t)$) and t are not bisimilar in general, as we can see in the following result.

Proposition 6. *We have $\mathcal{S}k.k v \not\approx v$.*

The \xrightarrow{E} transition from $\mathcal{S}k.k v$ cannot be matched by v . In terms of contextual equivalence, it is not possible to equate a stuck term and a value (it is also forbidden by the relation \approx_c^1 of Section 2.3). The CPS equivalence cannot distinguish between stuck terms and values, because the CPS translation turns all λ_S terms into λ -calculus terms of the form $\lambda k_1 k_2. t$, where k_1 is the continuation up to the first enclosing reset, and k_2 is the continuation beyond this reset. Therefore, the CPS translation (and CPS equivalence) assumes that there is always an enclosing reset, while contextual equivalence does not. To be in accordance with CPS, the contextual equivalence should be changed, so that it tests terms only in contexts with an outermost delimiter. We conjecture that the CPS equivalence is included in such a modified contextual equivalence. Note that stuck terms can no longer be observed in such modified relation, because a term within a reset cannot become stuck (see the proof of Proposition 3). Therefore, the bisimilarity of this article is too discriminative w.r.t. to this modified equivalence, and a new complete bisimilarity has to be found.

Conversely, there exist bisimilar terms that are not CPS equivalent:

Proposition 7. *1. We have $\Omega \approx \Omega \Omega$, but Ω and $\Omega \Omega$ are not CPS equivalent.
2. Let $\Theta = \theta \theta$, where $\theta = \lambda x y. y (\lambda z. x x y z)$, and $\Delta = \lambda x. \delta_x \delta_x$, where $\delta_x = \lambda y. x (\lambda z. y y z)$. We have $\Theta \approx \Delta$, but Θ and Δ are not CPS equivalent.*

Contextual equivalence puts all diverging terms in one equivalence class, while CPS equivalence is more discriminating. Furthermore, as is usual with equational theories for λ -calculi, CPS equivalence is not strong enough to equate Turing's and Curry's (call-by-value) fixed point combinators.

5 Conclusion

In this article, we propose a first study of the behavioral theory of a λ -calculus with delimited-control operators. We discuss various definitions of contextual equivalence, and we define an LTS-based applicative bisimilarity which is sound and complete w.r.t. the chosen contextual equivalence. Finally, we point out some differences between bisimilarity and CPS equivalence. We believe this work can be pursued in the following directions.

Up-to techniques. Up-to techniques [23,13,22] have been introduced to simplify the use of bisimulation in proofs of program equivalences. The idea is to prove terms equivalences using relations that are usually not bisimulations, but are included in bisimulations. The validity of some applicative bisimilarities up-to context remains an open problem in the λ -calculus [13]; nevertheless, we want to see if some up-to techniques can be applied to the bisimulations of this article.

Other forms of bisimilarity. Applicative bisimilarity is simpler to prove than contextual equivalence, but its definition still involves some quantification over values and pure contexts in labels. *Normal bisimilarities* are easier to use because their definitions do not feature such quantification. Lassen has developed a notion of normal bisimilarity, sound in various λ -calculi [14,15,16], and also complete in the $\lambda\mu$ -calculus with state [24]. It would be interesting to see if this equivalence can be defined in a calculus with delimited control, and if it is complete in this setting. Another kind of equivalence worth exploring is *environmental bisimilarity* [22].

Other calculi with control. Defining an applicative bisimilarity for the call-by-name variant of λ_S and for the hierarchy of delimited-control operators [7] should be straightforward. We plan to investigate applicative bisimilarities for a typed λ_S as well [4]. The problem seems more complex in calculi with abortive operators, such as call/cc. Because there is no delimiter for capture, these languages are not compositional (i.e., $t \rightarrow_v t'$ does not imply $E[t] \rightarrow_v E[t']$), which makes the definition of a compositional LTS more difficult.

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A Contextual Equivalence

We prove the result admitted in Section 2.3.

Lemma 8. *Let $t_0 = (Sk.\lambda x.x) \Omega$. Let t be such that x is the only free variable of t . If $t\{t_0/x\} \Downarrow_v v_0$, then there exists v such that $v_0 = v\{t_0/x\}$, and for all t_1 , we have $t\{t_1/x\} \Downarrow_v v\{t_1/x\}$.*

Proof. We proceed by induction on the number n of reduction steps in $t\{t_0/x\} \Downarrow_v v_0$. If $n = 0$, then t is a value, and the result is obvious. For $n > 0$, we have $t\{t_0/x\} \rightarrow_v t' \Downarrow_v v_0$. By case analysis on the reduction $t\{t_0/x\} \rightarrow_v t'$, we can see that t' can be written $t''\{t_0/x\}$. In particular, the reduction $F[\langle t_0 \rangle] \rightarrow_v F[\langle (\lambda x.\langle x \Omega \rangle) (\lambda x.x) \rangle]$ is not possible, because otherwise t' would diverge. By induction hypothesis, there exists v such that $v_0 = v\{t_0/x\}$, and for all t_1 , we have $t''\{t_1/x\} \Downarrow_v v\{t_1/x\}$. Therefore, $t\{t_1/x\} \Downarrow_v v\{t_1/x\}$ holds, as wished. \square

B Bisimilarity

B.1 Labelled Transition System

Lemma 9. *If $t \xrightarrow{E} t'$, then there exist E' , k , and s such that $t = E'[Sk.s]$ and $t' = \langle s\{\lambda x.\langle E[E'[x]] \rangle/k \} \rangle$.*

Proof. By induction on $t \xrightarrow{E} t'$. Suppose we have $Sk.s \xrightarrow{E} s\{\lambda x.\langle E[x] \rangle/k \}$; the result is obvious. Suppose we have $t = v t_0$ and $t_0 \xrightarrow{v E} t'$. By induction hypothesis there exists E' , s such that $t_0 = E'[Sk.s]$ and $t' = \langle s\{\lambda x.\langle (v E)[E'[x]] \rangle/k \} \rangle$. Therefore we have $v t_0 = v E'[Sk.s]$, and $t' = \langle s\{\lambda x.\langle E[(v E'[x])] \rangle/k \} \rangle$, as required. The other cases are treated similarly. \square

Lemma 10 (Lemma 3 in the article). *The following holds:*

- We have $\xrightarrow{\tau} = \rightarrow_v$.
- If $t \xrightarrow{E} t'$, then t is a stuck term, and $\langle E[t] \rangle \xrightarrow{\tau} t'$.
- If $t \xrightarrow{v} t'$, then t is a value, and $t v \xrightarrow{\tau} t'$.

Proof. For the first result, the only difficult transition to check is the capture by shift. If $\langle t \rangle \xrightarrow{\tau} t'$ with $t \xrightarrow{\square} t'$, then by Lemma 9, there exists E , s such that $t = E[Sk.s]$ and $t' = \langle s\{\lambda x.\langle E[x] \rangle/k \} \rangle$. We have $\langle E[Sk.s] \rangle \rightarrow_v \langle s\{\lambda x.\langle E[x] \rangle/k \} \rangle$, as wished. Conversely, if $F[\langle E[Sk.s] \rangle] \rightarrow_v F[\langle s\{\lambda x.\langle E[x] \rangle/k \} \rangle]$, then one can check that $F[\langle E[Sk.s] \rangle] \xrightarrow{\tau} F[\langle s\{\lambda x.\langle E[x] \rangle/k \} \rangle]$ holds by induction on F .

For the second result, by Lemma 9, there exist E' , k and s such that $t = E'[Sk.s]$ and $t' = \langle s\{\lambda x.\langle E[E'[x]] \rangle/k \} \rangle$. The term t is stuck by Lemma 1. We can also easily check that $\langle E[t] \rangle = \langle E[E'[Sk.s]] \rangle \xrightarrow{\tau} \langle s\{\lambda x.\langle E[E'[x]] \rangle/k \} \rangle$ holds.

For the last result, by looking at the LTS, it is easy to see that t must be a value $\lambda x.s$, and $t' = s\{v/x\}$, and $t v \xrightarrow{\tau} t'$. \square

B.2 Howe's method

Lemma 11. *If $t_0 \approx^\bullet t_1$, then there exists a substitution σ which closes t_0 and t_1 such that $t_0\sigma (\approx^\bullet)^c t_1\sigma$, and the size of the derivation of $t_0\sigma (\approx^\bullet)^c t_1\sigma$ is equal to the size of the derivation of $t_0 \approx^\bullet t_1$.*

Proof. By induction on $t_0 \approx^\bullet t_1$. Suppose we have $t_0 \approx^\circ t_1$. Let σ be a substitution which closes t_0 and t_1 ; we have $t_0\sigma \approx^\circ t_1\sigma$. The remaining cases are easy using induction. \square

Lemma 12. *Let t be a closed term. If $t \xrightarrow{\alpha} t'$, then for all $\alpha (\approx^\bullet)^c \alpha'$, there exists t'' such that $t \xrightarrow{\alpha'} t''$ and $t' (\approx^\bullet)^c t''$.*

Proof. By induction on $t \xrightarrow{\alpha} t'$. \square

Lemma 13. *If $\lambda x.t_0 (\approx^\bullet)^c \lambda x.t_1$ and $v_0 (\approx^\bullet)^c v_1$ then $t_0\{v_0/x\} (\approx^\bullet)^c t_1\{v_1/x\}$.*

Proof. By induction on $\lambda x.t_0 (\approx^\bullet)^c \lambda x.t_1$.

Suppose $\lambda x.t_0 \approx \lambda x.t_1$. We have $\lambda x.t_0 \xrightarrow{v_0} t_0\{v_0/x\}$, so by Lemma 12, there exists t'_0 such that $\lambda x.t_0 \xrightarrow{v_1} t'_0$ and $t_0\{v_0/x\} \approx^\bullet t'_0$. By bisimilarity, there exists t'_1 such that $\lambda x.t_1 \xrightarrow{v_1} t'_1$ and $t'_0 \approx t'_1$. The only possible outcome is $t'_1 = t_1\{v_1/x\}$, therefore we have $t_0\{v_0/x\} \approx^\bullet t_1\{v_1/x\}$. Because the considered terms are closed, we have the required result.

Suppose $\lambda x.t_0 \approx^\bullet \lambda x.t_1$. The result can easily be proved using induction and a similar reasoning as in the first case.

Suppose $\lambda x.t_0 \widehat{\approx}^\bullet \lambda x.t_1$. Then we have $t_0 \approx^\bullet t_1$; therefore, by Lemma 6, we have $t_0\{v_0/x\} \approx^\bullet t_1\{v_1/x\}$, as required. \square

Lemma 14. *If $v_0 (\approx^\bullet)^c t_1$, then there exists v_1 such that $t_1 \xrightarrow{\tau} v_1$ and $v_0 (\approx^\bullet)^c v_1$.*

Proof. By induction on $v_0 (\approx^\bullet)^c t_1$.

Suppose $v_0 \approx t_1$. Let $v_0 = \lambda x.t_0$; for all v , we have $v_0 \xrightarrow{v} t_0\{v/x\}$. By bisimilarity, there exists t_1^v such that $v_0 \xrightarrow{v} t_1^v$ and $t_0\{v/x\} \approx t_1^v$. Therefore there exists $v_1 = \lambda x.t_1'$ such that $t_1 \xrightarrow{\tau} v_1 \xrightarrow{v} t_1'\{v/x\}$ and $t_0\{v/x\} \approx t_1'\{v/x\}$. Because this holds for all v , we have $t_0 \approx^\circ t_1'$, therefore we have $t_0 \approx^\bullet t_1'$. From this observation, we deduce $\lambda x.t_0 (\approx^\bullet)^c \lambda x.t_1'$, as wished.

The case $v_0 \approx^\bullet t_1$ relies on induction and a similar reasoning as in the first case. Suppose $v_0 \widehat{\approx}^\bullet t_1$. Let $v_0 = \lambda x.t_0$; we have $t_1 = \lambda x.t_1'$ with $t_0 \approx^\bullet t_1'$. Because \approx^\bullet is a congruence, we have $\lambda x.t_0 \approx^\bullet \lambda x.t_1'$, as wished. \square

Lemma 15. *If $E_0 \approx^\bullet E_1$ and $t_0 \approx^\bullet t_1$ then $E_0[t_0] \approx^\bullet E_1[t_1]$.*

Proof. By induction on $E_0 \approx^\bullet E_1$. \square

Lemma 16 (Lemma 7 in the article). *If $t_0 (\approx^\bullet)^c t_1$ and $t_0 \xrightarrow{\alpha} t'_0$, then for all $\alpha (\approx^\bullet)^c \alpha'$, there exists t'_1 such that $t_1 \xrightarrow{\alpha'} t'_1$ and $t'_0 (\approx^\bullet)^c t'_1$.*

Proof. By induction on the size of the derivation of $t_0 (\approx^\bullet)^c t_1$.

If $t_0 \approx^\circ t_1$, then we have $t_0 \approx t_1$ because we work with closed terms. By Lemma 12, there exists t_0'' such that $t_0 \xrightarrow{\alpha'} t_0''$ and $t_0' (\approx^\bullet)^c t_0''$. By bisimilarity, there exists t_1' such that $t_1 \xRightarrow{\alpha'} t_1'$ and $t_0'' \approx t_1'$ (i.e., $t_0'' \approx^\circ t_1'$ because the terms are closed). Therefore we have $t_0' (\approx^\bullet)^c \approx^\circ t_1'$, i.e., $t_0' (\approx^\bullet)^c t_1'$, as required.

If $t_0 \approx^\bullet t_2 \approx^\circ t_1$, then by Lemma 11, there exists σ such that $t_0 \sigma (\approx^\bullet)^c t_2 \sigma$ and the size of the derivation of $t_0 \sigma (\approx^\bullet)^c t_2 \sigma$ is the same as for $t_0 (\approx^\bullet)^c t_2$. Because t_0 and t_1 are closed, and by definition of \approx° , we have in fact $t_0 (\approx^\bullet)^c t_2 \sigma \approx t_1$. By induction hypothesis, there exists t_2' such that $t_2 \sigma \xRightarrow{\alpha'} t_2'$ and $t_0' (\approx^\bullet)^c t_2'$. By bisimilarity, there exists t_1' such that $t_1 \xRightarrow{\alpha'} t_1'$ and $t_2' \approx t_1'$ (i.e., $t_2' \approx^\circ t_1'$ because the terms are closed). Therefore we have $t_0' (\approx^\bullet)^c \approx^\circ t_1'$, i.e., $t_0' (\approx^\bullet)^c t_1'$, as required.

If $t_0 \widehat{\approx^\bullet} t_1$, then we distinguish several cases, depending on the outermost operator.

Suppose $t_0 = \lambda x.s_0$ and $t_1 = \lambda x.s_1$ with $s_0 \approx^\bullet s_1$. The only possible transition is $t_0 \xrightarrow{v} s_0\{v/x\}$. We have $t_1 \xrightarrow{v'} s_1\{v'/x\}$. By Lemma 6, we have $s_0\{v/x\} \approx^\bullet s_1\{v'/x\}$, and because x is the only free variable of s_0 and s_1 , we have $s_0\{v/x\} (\approx^\bullet)^c s_1\{v'/x\}$, as required.

Suppose $t_0 = \mathcal{S}k.s_0$ and $t_1 = \mathcal{S}k.s_1$ with $s_0 \approx^\bullet s_1$. The only possible transition is $t_0 \xrightarrow{E} \langle s_0\{\lambda x.\langle E[x]\rangle/k\} \rangle$. We have $t_1 \xrightarrow{E'} \langle s_1\{\lambda x.\langle E'[x]\rangle/k\} \rangle$. Because \approx^\bullet is a congruence and by Lemma 15, we have $\lambda x.\langle E[x]\rangle (\approx^\bullet)^c \lambda x.\langle E'[x]\rangle$. Therefore, by Lemma 6, we have $\langle s_0\{\lambda x.\langle E[x]\rangle/k\} \rangle \approx^\bullet \langle s_1\{\lambda x.\langle E'[x]\rangle/k\} \rangle$, and because k is the only free variable of s_0 and s_1 , we have $\langle s_0\{\lambda x.\langle E[x]\rangle/k\} \rangle (\approx^\bullet)^c \langle s_1\{\lambda x.\langle E'[x]\rangle/k\} \rangle$, as required.

Suppose $t_0 = (\lambda x.s_0) v_0$ and $t_1 = t_1^1 t_1^2$ with $\lambda x.s_0 (\approx^\bullet)^c t_1^1$ and $v_0 (\approx^\bullet)^c t_1^2$. The only possible transition is $t_0 \xrightarrow{\tau} s_0\{v_0/x\}$. By Lemma 14, there exists $\lambda x.s_1, v_1$ such that $t_1^1 \xRightarrow{\tau} \lambda x.s_1$, $t_1^2 \xRightarrow{\tau} v_1$, $\lambda x.s_0 (\approx^\bullet)^c \lambda x.s_1$, and $v_0 (\approx^\bullet)^c v_1$. From $t_1^1 \xRightarrow{\tau} \lambda x.s_1$ and $t_1^2 \xRightarrow{\tau} v_1$, we can deduce $t_1 \xRightarrow{\tau} s_1\{v_1/x\}$, and from $\lambda x.s_0 (\approx^\bullet)^c \lambda x.s_1$ and $v_0 (\approx^\bullet)^c v_1$, we have $s_0\{v_0/x\} \approx^\bullet s_1\{v_1/x\}$ by Lemma 13. Hence, we have the required result.

Suppose $t_0 = v_0 s_0$ and $t_1 = t_1^1 s_1$ with $v_0 (\approx^\bullet)^c t_1^1$ and $s_0 (\approx^\bullet)^c s_1$. The only possible transition is $t_0 \xrightarrow{\tau} v_0 s_0'$, where $s_0 \xrightarrow{\tau} s_0'$. By Lemma 14, there exists v_1 such that $t_1^1 \xRightarrow{\tau} v_1$ and $v_0 (\approx^\bullet)^c v_1$. By induction hypothesis, there exists s_1' such that $s_1 \xRightarrow{\tau} s_1'$ and $s_1 (\approx^\bullet)^c s_1'$. Therefore we have $t_1 \xRightarrow{\tau} v_1 s_1'$, and because $(\approx^\bullet)^c$ is a congruence, we have $v_0 s_0' (\approx^\bullet)^c v_1 s_1'$, hence the result holds.

Suppose $t_0 = t_0^1 t_0^2$ and $t_1 = t_1^1 t_1^2$ with $t_0^1 (\approx^\bullet)^c t_1^1$ and $t_0^2 (\approx^\bullet)^c t_1^2$. The only possible transition is $t_0 \xrightarrow{\tau} t_0^1 t_0^2$, where $t_0^1 \xrightarrow{\tau} t_0^1'$. By induction, there exists t_1^1' such that $t_1^1 \xRightarrow{\tau} t_1^1'$ and $t_0^1 (\approx^\bullet)^c t_1^1'$. Therefore we have $t_1 \xRightarrow{\tau} t_1^1' t_1^2$, and because $(\approx^\bullet)^c$ is a congruence, we have $t_0^1 t_0^2 (\approx^\bullet)^c t_1^1' t_1^2$, hence the result holds. The case $t_0 = \langle s_0 \rangle$, where s_0 is not a value, is treated similarly.

Suppose $t_0 = \langle v_0 \rangle$ and $t_1 = \langle s_1 \rangle$ with $v_0 (\approx^\bullet)^c s_1$. The only possible transition is $t_0 \xrightarrow{\tau} v_0$. By Lemma 14, there exists v_1 such that $s_1 \xrightarrow{\tau} v_1$ and $v_0 (\approx^\bullet)^c v_1$. We have $t_1 \xrightarrow{\tau} v_1$, with $v_0 (\approx^\bullet)^c v_1$, hence the result holds. \square

Lemma 17. *The relation $((\approx^\bullet)^c)^*$ is a bisimulation.*

Proof. We know that $((\approx^\bullet)^c)^*$ is symmetric by Lemma 6, so it is enough to prove that $((\approx^\bullet)^c)^*$ is a simulation. Let $t_0((\approx^\bullet)^c)^*t_1$; there exists an integer k such that $t_0((\approx^\bullet)^c)^k t_1$. Let $(t_0^i)_{i \in \{1 \dots k\}}$ be the terms such that $t_0(\approx^\bullet)^c t_0^1(\approx^\bullet)^c t_0^2 \dots t_0^{k-1}(\approx^\bullet)^c t_0^k = t_1$. Let $t_0 \xrightarrow{\alpha} t_0'$. We prove by induction on $i \in 1 \dots k$ that there exists $t_0^{i'}$ such that $t_0' \xrightarrow{\alpha} t_0^{i'}$ and $t_0'((\approx^\bullet)^c)^i t_0^{i'}$. Suppose $i = 1$. We have $\alpha(\approx^\bullet)^c \alpha$ so by Lemma 16, there exists $t_0^{1'}$ such that $t_0' \xrightarrow{\alpha} t_0^{1'}$ and $t_0'(\approx^\bullet)^c t_0^{1'}$, as wished. The case $1 < i \leq k$ is easy by induction. In particular, for $i = k$, we have $t_1 = t_0^k \xrightarrow{\alpha} t_0^{k'}$ and $t_0'((\approx^\bullet)^c)^k t_0^{k'}$. We have the required result because $((\approx^\bullet)^c)^k \subseteq ((\approx^\bullet)^c)^*$. \square

Theorem 5 (Theorem 2 in the article). *We have $\approx \subseteq \approx_c$.*

Proof. Let $t_0 \approx t_1$, and C a context. Because \approx is a congruence, we have $C[t_0] \approx C[t_1]$. If $C[t_0] \Downarrow_v v_0$, then we have $C[t_0] \xrightarrow{\tau} v_0 \xrightarrow{v}$; by bisimilarity, there exists v_1 such that $C[t_1] \xrightarrow{\tau} v_1 \xrightarrow{v}$, therefore we have $C[t_1] \Downarrow_v v_1$. The reasoning is the same if $C[t_0] \Downarrow_v t_0'$, where t_0' is stuck, and for the evaluations of $C[t_1]$. \square

B.3 Completeness

Theorem 6 (Theorem 3 in the article). *We have $\dot{\approx}_c \subseteq \approx$.*

Proof. Because $\dot{\approx}_c$ is symmetric, it is enough to prove that $\dot{\approx}_c$ is a big-step simulation. Let $t_0 \dot{\approx}_c t_1$. We have two cases to consider.

Suppose $t_0 \xrightarrow{v} t_0'$. Then we have $t_0 \Downarrow_v v_0$ for some v_0 . By definition of $\dot{\approx}_c$, there exists v_1 such that $t_1 \Downarrow_v v_1$. Therefore, we have $t_1 \xrightarrow{v} t_1'$ for some t_1' and $t_0 v \xrightarrow{\tau} t_0'$ and $t_1 v \xrightarrow{\tau} t_1'$ by Lemma 10. Hence, we have $t_0 v \approx t_0'$ and $t_1 v \approx t_1'$ by Lemma 4. Because $t_0 \dot{\approx}_c t_1$, we have $t_0 v \dot{\approx}_c t_1 v$. Finally, we have $t_0' \approx t_0 v \dot{\approx}_c t_1 v \approx t_1'$ and $t_0' \dot{\approx}_c t_1'$ by Theorem 5 and transitivity of $\dot{\approx}_c$.

Suppose $t_0 \xrightarrow{E} t_0'$. Then we have $t_0 \Downarrow_v t_0''$ for some t_0'' . By definition of $\dot{\approx}_c$, there exists t_1'' such that $t_1 \Downarrow_v t_1''$. Therefore, by definition of the LTS, we have $t_1 \xrightarrow{E} t_1'$ for some t_1' . We then have $\langle E[t_0] \rangle \xrightarrow{\tau} t_0'$ and $\langle E[t_1] \rangle \xrightarrow{\tau} t_1'$ by Lemma 10. Hence, we have $\langle E[t_0] \rangle \approx t_0'$ and $\langle E[t_1] \rangle \approx t_1'$ by Lemma 4. Because $t_0 \dot{\approx}_c t_1$, we have $\langle E[t_0] \rangle \dot{\approx}_c \langle E[t_1] \rangle$. Finally, we have $t_0' \approx \langle E[t_0] \rangle \dot{\approx}_c \langle E[t_1] \rangle \approx t_1'$, and, therefore we have $t_0' \dot{\approx}_c t_1'$ by Theorem 5 and transitivity of $\dot{\approx}_c$. \square

C Proof of Proposition 5

Proof. Let E_0 be such that $\text{fv}(E_0) = \emptyset$. We let $\sigma_i^{E_0}$ (resp. $\sigma_i^{\lambda x. E_0[x]}$) range over substitutions mapping k_i to $\lambda y. \langle E_1^i[E_0[E_2^i[y]]] \rangle$ (resp. $\lambda y. \langle E_1^i[(\lambda x. E_0[x]) E_2^i[y]] \rangle$)

for some fixed E_1^i, E_2^i . We define

$$\begin{aligned}\mathcal{R}_1 &= \{(F[(\lambda x.E_0[x])\ t]\sigma_0^{\lambda x.E_0[x]} \dots \sigma_n^{\lambda x.E_0[x]}, F[E_0[t]]\sigma_0^{E_0} \dots \sigma_n^{E_0}), \\ &\quad \text{fv}(t, F) \subseteq \{k_0 \dots k_n\}\} \\ \mathcal{R}_2 &= \{(t\sigma_0^{\lambda x.E_0[x]} \dots \sigma_n^{\lambda x.E_0[x]}, t\sigma_0^{E_0} \dots \sigma_n^{E_0}), \text{fv}(t) \subseteq \{k_0 \dots k_n\}\}\end{aligned}$$

and we prove that $\mathcal{R}_1 \cup \mathcal{R}_2$ is a bisimulation. First, let $t_0 \mathcal{R}_1 t_1$ with $t_0 = F[(\lambda x.E_0[x])\ t]\sigma_0^{\lambda x.E_0[x]} \dots \sigma_n^{\lambda x.E_0[x]}$ and $t_1 = F[E_0[t]]\sigma_0^{E_0} \dots \sigma_n^{E_0}$; we consider the possible transitions from t_0 .

- If $t \xrightarrow{\tau} t'$, then $t_0 \xrightarrow{\tau} t'_0 = F[(\lambda x.E_0[x])\ t']\sigma_0^{\lambda x.E_0[x]} \dots \sigma_n^{\lambda x.E_0[x]}$ and $t_1 \xrightarrow{\tau} t'_1 = F[E_0[t']]\sigma_0^{E_0} \dots \sigma_n^{E_0}$. We still have $t'_0 \mathcal{R}_1 t'_1$.
- If t is a value v , then $t_0 \xrightarrow{\tau} t'_0 = F[E_0[v]]\sigma_0^{\lambda x.E_0[x]} \dots \sigma_n^{\lambda x.E_0[x]}$ (with rule (β_v)), and $t_1 = F[E_0[v]]\sigma_0^{E_0} \dots \sigma_n^{E_0}$. We have $t'_0 \mathcal{R}_2 t_1$.
- If t is a stuck term, then we have two possibilities. If F contains a reset, then the shift in t can capture the context up to this delimiter, resulting in $t_0 \xrightarrow{\tau} t'_0 = F'[t'\{\lambda y.\langle E_1[(\lambda x.E_0[x])\ E_2[y]]\rangle/k_{n+1}\}]\sigma_0^{\lambda x.E_0[x]} \dots \sigma_n^{\lambda x.E_0[x]}$ and $t_1 \xrightarrow{\tau} t'_1 = F'[t'\{\lambda y.\langle E_1[E_0[E_2[y]]]\rangle/k_{n+1}\}]\sigma_0^{E_0} \dots \sigma_n^{E_0}$. Otherwise, t can capture F , giving $t_0 \xrightarrow{E} t'_0 = t'\{\lambda y.\langle E_1[(\lambda x.E_0[x])\ E_2[y]]\rangle/k_{n+1}\}\sigma_0^{\lambda x.E_0[x]} \dots \sigma_n^{\lambda x.E_0[x]}$, and similarly $t_1 \xrightarrow{E} t'_1 = t'\{\lambda y.\langle E_1[E_0[E_2[y]]]\rangle/k_{n+1}\}\sigma_0^{E_0} \dots \sigma_n^{E_0}$. In both cases, we have $t'_0 \mathcal{R}_2 t'_1$.

Conversely, one can check that the transitions from t_1 are matched by t_0 . Now, we consider $t_0 \mathcal{R}_2 t_1$ with $t_0 = t\sigma_0^{\lambda x.E_0[x]} \dots \sigma_n^{\lambda x.E_0[x]}$ and $t_1 = t\sigma_0^{E_0} \dots \sigma_n^{E_0}$. We enumerate the possible transitions from t_0 .

- If $t \xrightarrow{\alpha} t'$, then $t_0 \xrightarrow{\alpha} t'_0 = t'\sigma_0^{\lambda x.E_0[x]} \dots \sigma_n^{\lambda x.E_0[x]}$ and $t_1 \xrightarrow{\alpha} t'_1 = t'\sigma_0^{E_0} \dots \sigma_n^{E_0}$. We still have $t'_0 \mathcal{R}_2 t'_1$.
- Suppose $t = F[(\lambda z.t')\ k_i]$. Then, using rule (β_v) , we have the transitions $t_0 \xrightarrow{\tau} t'_0 = F[t'\{\lambda y.\langle E_1^i[(\lambda x.E_0[x])\ E_2^i[y]]\rangle/z\}]\sigma_0^{\lambda x.E_0[x]} \dots \sigma_n^{\lambda x.E_0[x]}$ and $t_1 \xrightarrow{\tau} t'_1 = F[t'\{\lambda y.\langle E_1^i[E_0[E_2^i[y]]]\rangle/z\}]\sigma_0^{E_0} \dots \sigma_n^{E_0}$. The terms t'_0 and t'_1 can be written $F[t'\{k_i/z\}]\sigma_0^{\lambda x.E_0[x]} \dots \sigma_n^{\lambda x.E_0[x]}$ and $F[t'\{k_i/z\}]\sigma_0^{E_0} \dots \sigma_n^{E_0}$, therefore we have $t'_0 \mathcal{R}_2 t'_1$. The reasoning is the same if $t = F[\langle k_i \rangle]$.
- If $t = F[k_i\ v]$, then $t_0 \xrightarrow{\tau} t'_0 = F[\langle E_1^i[(\lambda x.E_0[x])\ E_2^i[v]] \rangle]\sigma_0^{\lambda x.E_0[x]} \dots \sigma_n^{\lambda x.E_0[x]}$, and $t_1 \xrightarrow{\tau} t'_1 = F[\langle E_1^i[E_0[E_2^i[v]]] \rangle]\sigma_0^{E_0} \dots \sigma_n^{E_0}$. We have $t'_0 \mathcal{R}_1 t'_1$.

Conversely, one can check that the transitions from t_1 are matched by t_0 . Finally, $\mathcal{R}_1 \cup \mathcal{R}_2$ is a bisimulation, as wished. \square